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Interactions between pairs of oblique waves on a mixing layer

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Abstract

We consider the weakly nonlinear spatial evolution of two pairs of oblique waves superimposed on an inviscid mixing layer, with each wave being slightly amplified on a linear basis. One pair of waves is assumed to be inclined at an angle θ_1 to the plane of the mixing layer, the other at an angle θ_2. A nonlinear critical layer analysis is employed to derive equations governing the evolution of the instability wave amplitudes, which contain a coupling between the pairs as well as within each pair. These equations are discussed and it is shown that, as in related work for other flows, these equations may develop a singularity at a finite distance downstream.

Keywords: Shear layer; Nonlinear stability; Oblique waves

1. Introduction

In a landmark paper, Goldstein and Choi [1] identified a critical layer mechanism by which oblique disturbances to a shear layer could undergo extremely rapid growth when the disturbance consisted of a pair of oblique waves at equal and opposite angles to the shear layer. This mechanism was similar to that for a single wave found earlier by Hickernell [2], and the amplitude equations arising from the theory of nonlinear non-equilibrium critical layers are sometimes called "Hickernell-type" equations. Later [3,4], it was found that the growth could be even more rapid if in addition to the pair of oblique waves, the disturbance included a plane wave. This approach required that the oblique waves be the subharmonic of the plane wave, meaning that they had to be inclined at $\pm 60°$ to the plane of the mean flow, and this form of disturbance is known as a (subharmonic) resonant triad. If the amplitude of the disturbance was $O(\varepsilon)$, the evolution of a purely planar disturbance would first become nonlinear on a length-scale (or time-scale for temporally evolving disturbances) of $O(\varepsilon^{-1/2})$ while a disturbance consisting of a pair of oblique waves would first experience nonlinear growth on the much shorter length-scale of $O(\varepsilon^{-1/3})$ and the length-scale for a resonant triad was $O(\varepsilon^{-1/4})$ for the parametric resonance stage. In all cases, the growth first became nonlinear inside the critical layer, which is the location at which the velocity of the base flow is equal to the phase speed of the disturbance, $u_0(y_c) = c$. Mathematically, the approach taken in these studies was to employ matched asymptotic expansions, with an "outer" expansion away from the critical layer and an "inner" expansion near the critical layer, where rescaled variables where introduced.

A number of studies followed for various flows (e.g., [5–12]), and it has been claimed (e.g., [10]) that some experiments (e.g., [13,14]) have provided at least partial confirmation of the theory. In each of these studies, the development of the disturbance can be broken down into three stages. Initially, when the disturbance is small, the growth of the disturbance is linear. Eventually, when the disturbance is large enough, the growth becomes nonlinear, and finally at some point it becomes explosive, with the solutions of the amplitude equations having a singularity at a finite time (or a finite distance downstream). Two studies [6,15] have explored how weak viscosity affects the development of the disturbance. These studies would seem to indicate that viscous effects modify the singular behavior slightly but it appears that at high Reynolds numbers the result is still meaningful. Wu [6] showed that viscosity delays the occurrence of the finite distance singularity but does not appear to be

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able to eliminate it. Since a full discussion of the effects of viscosity is beyond the scope of this article, the interested reader is referred to [6,15].

Most of the early studies considered flows with only one critical layer, although Wu's study [4] of the Stokes layer allowed the possibility of more than one critical layer and several studies [7,11,12,16] have considered plane wakes and jets with two critical layers and interactions between different neutral modes.

In a sense, both the resonant triad studies and the wake studies can be thought of as extensions to the work of Goldstein and Choi [1], in that the Goldstein–Choi equation can be recovered from those other studies. In the present study, we are exploring another extension to [1]. We shall consider the instability of a mixing layer to pairs of oblique waves. However, while [1] and subsequent authors have considered only a single pair of oblique waves at an angle $\pm\theta$ to the plane of the base flow, or in [12] two pairs of oblique waves at the same angle but with different wave numbers, we shall consider the case of two pairs of oblique waves with the same wavenumber but at *different* angles, $\pm\theta_1$ and $\pm\theta_2$ with $\theta_1 \neq \theta_2$. Of course, the resonant triad studied in [3,4,9,10] is in a sense a special case of this with waves at 0° and $\pm 60^\circ$, and we should stress that our analysis is for two pairs of *oblique* waves and so is not applicable to the resonant triad which has only one pair of oblique waves together with a plane wave; the scalings for the resonant triad differ to those used here. Still another extension to [1], a pair of oblique waves at an angle other then $\pm 60^\circ$ together with a plane wave is currently being investigated [17]. Our reasons for exploring the present problem are to see if the addition of this second pair of waves leads to even faster growth than if only a single pair of waves was present. To the best of our knowledge, we are unaware of any experiments or numerical simulations that have explored this scenario, and we would suggest that it might be a worthwhile endeavor for people in those fields to pursue.

One more study which we should mention is the phase-locked interaction of Wu and Stewart [18], which in a sense fully demonstrates the power of nonlinear critical layers. Wu and Stewart were able to demonstrate that two waves superimposed on a shear layer would interact nonlinearly inside the critical layer, provided that they had (roughly) the same phase velocity, and it is their phase-locked mechanism which causes the nonlinear interaction presented here. In a way, the mechanism in [18] can be thought of as a "building block", and a number of other transition mechanisms can be constructed by superimposing several of the phase-locked modes.

Turning to an overview of our analysis, we will follow [1] and take the usual approach of posing an outer expansion in Section 2 and an inner expansion in Section 3 near the critical layer, where the outer expansion becomes disordered and singular. Matching these two expansions together, or more precisely, matching certain jumps across the critical layer, will yield our amplitude equations. In our analysis, we derive only one of the two amplitude equations, that for the pair of waves at $\pm\theta_1$; the equation for the other pair (at $\pm\theta_2$) follows from symmetry. The scaling used in the outer expansion is the usual $\varepsilon^{1/3}$ critical layer first used in [1], where ε is a non-dimensional parameter characterizing the amplitude of the disturbances; this scaling is used because it is the stage in which the amplitude equations first become nonlinear. This scaling means that there will be an $O(\varepsilon^{1/3})$ departure of the phase speed from its neutral value which in turn gives rise to a long length-scale $X = \varepsilon^{1/3}x$ on which the waves interact. We note that in our analysis, both pairs of waves may be of roughly the same size ($O(\varepsilon)$), or either of them may be smaller than this, or indeed, one or other of them may be absent altogether in which case the amplitude equation for the remaining pair of waves simply reduces to that of Goldstein and Choi [1]. One possibility allowed by this scaling then is that the one pair may be much larger than the other pair, which will still lead to very rapid growth of both pairs during the nonlinear stage. This last scenario might happen if one pair of waves were forced and the smaller pair were part of the background noise.

The structure of the remainder of the paper is as follows. In Section 2, we briefly sketch the flow in the outer region, posing a perturbation analysis outside the critical layer. Since the analysis so closely follows that of earlier studies, notably [1], the procedure will be only briefly sketched. In addition, in the outer expansion, only the linear terms are necessary for the derivation of our end result, and we need only present the analysis for one of the pairs, with the other pair following from symmetry. We should note here that, just as in earlier studies, it is necessary to include in our analysis a mean streamwise vortex motion and several other vortex motions both inside and outside the critical layer. Outside the critical layer, the x component of the velocity due to this streamwise vortex is as large ($O(\varepsilon)$) as the oblique waves which induce it, although the y and z components are smaller. The need for this streamwise vortex was first noted by [1], and indeed all the scales used in this paper follow exactly those of [1]. We will discuss these additional terms further in Section 3.

In Section 3, we sketch how to analyze the flow inside the critical layers, again omitting some of the details because of the similarity to earlier work [1], and arrive at the relevant jumps across the critical layers, which we match to the jumps from the outer expansion given in Section 2. Once again, we only present the analysis for one of the pairs, with that for the other pair following by symmetry. This leads us to the amplitude equations, which are a pair of coupled nonlinear integro-differential equations governing the amplitudes of the waves. As noted above, when one or other of the pairs of waves is absent, the equation for the remaining pair reduces to that of [1]. As now appears usual for equations of this type, these equations may develop a singularity at a finite distance downstream, the physical significance of which is still not completely understood. Finally, in Section 4, we make some concluding remarks.

2. Formulation and outer expansion

We consider the spatial stability of the tanh y mixing layer $\bar{u}(y) = 1 + \tanh y$ to perturbations of $O(\varepsilon)$ where $\varepsilon \ll 1$ is a dimensionless amplitude parameter. We consider oblique perturbations proportional to $\exp[i\alpha(\xi\cos\theta \pm z\sin\theta)]$, where $\xi = x - ct$. The initial perturbation consists of two pairs of oblique waves, one pair at an angle θ_1, the other θ_2, and in what follows, we shall use the notation $\hat{c}_1 = \cos\theta_1$, $\hat{s}_1 = \sin\theta_1$, and similarly for θ_2. Since we are interested in the spatial stability problem, the wave number will take the neutral value $\alpha = 1$, while the phase velocities of the waves are perturbed slightly from neutral, so that the pair of waves at angle θ_1 has a phase velocity of $c_1 = 1 + \mu\gamma_1$ while that at angle θ_2 has a phase velocity of $c_2 = 1 + \mu\gamma_2$; these perturbations to the phase velocities cause the disturbances to develop on the long length-scale $X = \mu x$, where $\mu \ll 1$ is a second small parameter. The balance between these two small parameters, ε and μ, determines which flow regime we are in. In the present study, we are interested in the stage of evolution in which nonlinear effects first become important, and by setting $\mu = \varepsilon^{1/3}$, the linear and nonlinear jumps across the critical layer which we present in the next section enter at the same order, leading to a nonlinear evolution equation.

Assuming the flow to be inviscid (which requires in practice that the Reynolds number $\mathrm{Re} \gg \varepsilon^{-1}$) and incompressible, the equations of motion can be written in non-dimensional form as

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\nabla p, \qquad \nabla\cdot\underline{u} = 0, \tag{1}$$

where the velocity components $\underline{u} = (\bar{u} + \varepsilon\tilde{u}, \varepsilon\tilde{v}, \varepsilon\widetilde{w})$ and the perturbation pressure $p = \varepsilon\tilde{p}$ are expanded as $\tilde{u} = u^{(1)} + \varepsilon^{1/3}u^{(2)} + \varepsilon^{2/3}u^{(3)} + \cdots$ with similar expressions for $\tilde{v}$, $\widetilde{w}$ and $\tilde{p}$. The basic pressure is a constant which can be set to zero without any loss of generality. The lowest order disturbance is composed of two pairs of oblique waves, with each pair of equal amplitude at equal and opposite angles to the mean flow. Since $\alpha = 1$, the vertical velocity at lowest order can be written as

$$v^{(1)} = 2\left[\hat{v}_1^{(1)}A_1\,e^{i\hat{c}_1\xi_1}\cos\hat{s}_1 z + \hat{v}_2^{(1)}A_2\,e^{i\hat{c}_2\xi_2}\cos\hat{s}_2 z + c.c.\right], \tag{2}$$

where A_1 and A_2 are functions of the long length-scale X, '$c.c.$' means complex conjugate, and the phases of the waves are $\xi_1 = x - c_1 t$ and $\xi_1 = x - c_2 t$.

In the remainder of this section, we need only consider linear terms. Therefore, we shall give the results only for the waves at $\pm\theta_1$; the results for the waves at $\pm\theta_2$ follow from simply replacing the subscript "1" by the subscript "2".

Returning to the outer expansion, clearly, $\hat{v}_1^{(1)} = \operatorname{sech} y$ is a solution of the Rayleigh equation,

$$L\hat{v} = \hat{v}_{yy} + \left(2\operatorname{sech}^2 y - 1\right)\hat{v} = 0, \tag{3}$$

which vanishes as $y \to \pm\infty$. The other components at this order are

$$\hat{u}_1^{(1)} = \frac{i(1 - \hat{c}_1^2\cosh^2 y)}{\hat{c}_1\cosh^2 y\sinh y}, \tag{4}$$

$\hat{w}_1^{(1)} = -i\hat{s}_1 \operatorname{cosech} y$ and $\hat{p}_1^{(1)} = i\hat{c}_1 \operatorname{sech} y$. The $O(\varepsilon)$ streamwise velocity will also include a spanwise mean flow component induced by the flow inside the critical layer. This was first pointed out in [1] for the problem of a single pair of oblique waves.

At the next order, we find that the $O(\varepsilon^{4/3})$ vertical velocity $\hat{v}_1^{(2)}$ for the oblique waves at $\pm\theta_1$ obeys the following equation

$$L\hat{v}_1^{(2)} = -\frac{2\gamma_1 A_1}{\cosh^2 y\sinh y} - \frac{iA_1'(2 + \hat{c}_1^2\sinh 2y)}{\cosh^2 y\sinh y\hat{c}_1}, \tag{5}$$

where the left-hand side is the Rayleigh operator from (3) above. This equation (5) has a solution of the form

$$\hat{v}_1^{(2)} = C_1\hat{v}_1^{(1)} + D_1^{\pm}\hat{v}_1^{(1a)} - iA_1'\hat{c}_1\cosh y + \left(\gamma_1 A_1 + \frac{iA_1'}{\hat{c}_1}\right)\left(2\hat{v}_1^{(1)}\int^y \frac{y\,dy}{\sinh 2y} - \hat{v}_1^{(1a)}\ln\tanh|y|\right), \tag{6}$$

where $\hat{v}_1^{(1a)} = y\operatorname{sech} y + \sinh y$ is the second solution to the Rayleigh equation (3) above. The superscript ($\pm$) on D_1 refers to the regions above and below the critical layers respectively. Imposing the homogeneous boundary conditions as $y \to \pm\infty$ leads to a jump in the $D_1^{(1)}$ given by

$$D_1^+ - D_1^- = 2iA_1'\hat{c}_1. \tag{7}$$

It should be noted that the outer expansion is purely linear; nonlinear terms will appear at $O(\varepsilon^2)$ in the outer, but because the outer jumps appear much earlier than this, at $O(\varepsilon^{4/3})$, we do not need to calculate the nonlinear terms. Because of this the outer expansion for each pair of oblique waves is the same as that for a single pair of oblique waves.

In the present section, we have found expressions for the jumps across the critical layer from the outer expansion. In the next section, we will sketch how to find a second expression for these jumps from the solution inside critical layer solution; matching these jumps will lead to the amplitude equations governing the spatial evolution of A_1 and A_2. As we mentioned above, by setting $\mu = \varepsilon^{1/3}$, the linear and nonlinear jumps in the inner expansion will enter at the same order, leading to nonlinear amplitude equations.

3. Critical layer analysis

In order to obtain evolution equations for A_1 and A_2, we shall now pose inner expansions in the critical layer, where the outer expansion becomes disordered, and obtain expressions for the jumps across the critical layer. The details will again be largely omitted, because the analysis so closely parallels that of earlier work, and as in the outer expansion, we will derive the amplitude equation for the waves at $\pm\theta_1$, with the equation for the $\pm\theta_2$ waves following simply by interchanging the subscripts "1" and "2".

Near the critical layer at $y = 0$, we introduce the rescaled inner variable $Y = \mu^{-1} y$, where μ was the departure of the phase velocity from neutral. By setting $\mu = \varepsilon^{1/3}$, where of course ε was the order of magnitude of the disturbance in the outer expansion, the linear and nonlinear jumps in this section will enter at the same order, and with this scaling, our inner variables become $Y = \varepsilon^{-1/3} y$, $U = \varepsilon^{-1/3}(u-1)$, $V = \varepsilon^{-2/3} v$, $W = \varepsilon^{-1/3} w$ and $P = \varepsilon^{-4/3} p$.

These scalings are substituted into the governing (Euler) equations. The form of the inner expansion can be deduced from writing the outer solution in inner variables,

$$\begin{aligned} U &= Y + \varepsilon^{1/3} U_1 + \varepsilon^{2/3}\left(U_2 - \frac{Y^3}{3}\right) + \cdots, \\ V &= \varepsilon^{1/3} V_1 + \varepsilon^{2/3} V_2 + \cdots, \\ W &= \varepsilon^{1/3} W_1 + \varepsilon^{2/3} W_2 + \cdots, \\ P &= \varepsilon^{-1/3} P_{-1} + P_0 + \varepsilon^{1/3} P_1 + \varepsilon^{2/3} P_2 + \cdots. \end{aligned} \tag{8}$$

The streamwise velocity perturbation at lowest order can be written

$$U_1 = 2\left(U_1^{(1)} \mathrm{e}^{\mathrm{i}\hat{c}_1\xi_1} \cos \hat{s}_1 z + U_2^{(1)} \mathrm{e}^{\mathrm{i}\hat{c}_2\xi_2} \cos \hat{s}_2 z + c.c.\right), \tag{9}$$

where

$$U_1^{(1)} = -\hat{s}_1^2 \int_{-\infty}^{X} A_1(X_0)\, \mathrm{e}^{-\mathrm{i}\tilde{Y}_1(X - X_0)}\, \mathrm{d}X_0, \tag{10}$$

with a similar expression for $U_2^{(1)}$, and where $\widetilde{Y}_1 = (Y - \gamma_1)\hat{c}_1$ and $\widetilde{Y}_2 = (Y - \gamma_1)\hat{c}_2$. Likewise, for the other velocity and pressure components we find $V_1^{(1)} = A_1$, $W_1^{(1)} = -\hat{c}_1 U_1^{(1)}/\hat{s}_1$, $P_1^{(-1)} = \mathrm{i}\hat{c}_1 A_1$, $P_1^{(0)} = \hat{s}_1^2 A_1' + \mathrm{i}\hat{c}_1 C_1^{(1)}$ and

$$P_1^{(1)} = Y\left[\mathrm{i}A_1\hat{c}_1\left(\gamma_1 - \frac{Y}{2}\right) - A_1'\hat{s}_1^2\right] + p_1^{(1)}(X). \tag{11}$$

These linear disturbance components are of course identical to those where the disturbance contains a single pair of waves. At the next order, we find that in addition to the linear terms there are a number of nonlinear terms,

$$\begin{aligned} V_2 &= V_1^{(2)} \mathrm{e}^{\mathrm{i}(\hat{c}_1\xi_1 + \hat{s}_1 z)} + V_{22}^{(2)} \mathrm{e}^{2\mathrm{i}(\hat{c}_1\xi_1 + \hat{s}_1 z)} + V_{00}^{(2)} + V_{20}^{(2)} \mathrm{e}^{2\mathrm{i}\hat{c}_1\xi_1} + V_{02}^{(2)} \mathrm{e}^{2\mathrm{i}\hat{s}_1 z} \\ &\quad + \mathrm{e}^{\mathrm{i}(\hat{c}_1\xi_1 + \hat{c}_2\xi_2 + \hat{s}_1 z)}\left(V_A^{(2)} \mathrm{e}^{\mathrm{i}\hat{s}_2 z} + V_B^{(2)} \mathrm{e}^{-\mathrm{i}\hat{s}_2 z}\right) + \mathrm{e}^{\mathrm{i}(\hat{c}_1\xi_1 - \hat{c}_2\xi_2 + \hat{s}_1 z)}\left(V_C^{(2)} \mathrm{e}^{\mathrm{i}\hat{s}_2 z} + V_D^{(2)} \mathrm{e}^{-\mathrm{i}\hat{s}_2 z}\right), \end{aligned} \tag{12}$$

together with other terms that we do not need to calculate, since they do not affect the jumps across the critical layer. Once again, the linear terms are the same as those for a single pair of waves, with $V_1^{(2)} = C_1^{(1)} - i\hat{c}_1 A_1'$, and

$$\left[\frac{\partial}{\partial X} + \mathrm{i}\widetilde{Y}_1\right] U_1^{(2)} = \mathrm{i}Y\widetilde{Y}_1 U_1^{(1)} + \hat{s}_1^2\left(YA_1 - C_1^{(1)} - \mathrm{i}\hat{c}_1 A_1'\right), \tag{13}$$

and $W_1^{(2)} = \mathrm{i}\hat{c}_1 U_{1X}^{(1)}/\hat{s}_1 - U_1^{(2)}/\hat{s}_1$. For the nonlinear terms, there are both self-interaction terms and interactions between the θ_1 and θ_2 modes. Amongst the self-interaction terms, the plane wave term $(U_{20}^{(2)}, V_{20}^{(2)}, 0)$, the harmonic term $(U_{22}^{(2)}, 0, W_{22}^{(2)})$ and the cross-flow term $(U_{02}^{(2)}, V_{02}^{(2)}, W_{02}^{(2)})$ are each identical to the corresponding terms for the single pair of waves. The derivation of these terms may be found in [1], and the terms themselves are given in the appendix for completeness. The self-interaction terms also include a mean flow term $(U_{00}^{(2)}, 0, 0)$, and for this term it is necessary to include the contributions both from the self-interaction of the $\pm\theta_1$ terms and that from the $\pm\theta_2$ terms; each of these contributions is separately equal to the corresponding term for the single pair, and once again the derivation may be found in [1] while $U_{00}^{(2)}$ itself is given in the appendix.

We also have terms coming from the interactions between the modes. These terms were not present in [1], but are very similar to the sum and difference modes found in Wu and Stewart [18], and the $\mathrm{e}^{\mathrm{i}(\hat{c}_1\xi_1+\hat{c}_2\xi_2+(\hat{s}_1+\hat{s}_2)z)}$ term obeys

$$\begin{aligned}
&\left[\frac{\partial}{\partial X} + \mathrm{i}\big(\widetilde{Y}_1 + \widetilde{Y}_2\big)\right] V_{AYY}^{(2)} = \mathrm{i}\sin(\theta_2-\theta_1)\left(\frac{A_1 U_{2YY}^{(1)}}{\hat{s}_2} - \frac{A_2 U_{1YY}^{(1)}}{\hat{s}_1}\right) + \frac{2\sin^2(\theta_2-\theta_1)}{\hat{s}_2\hat{s}_1}\big(U_{2Y}^{(1)}U_1^{(1)} - U_2^{(1)}U_{1Y}^{(1)}\big),\\
&\left[\frac{\partial}{\partial X} + \mathrm{i}\big(\widetilde{Y}_1 + \widetilde{Y}_2\big)\right] U_{AX}^{(2)} = -V_A^{(2)} - A_1 U_{2Y}^{(1)} - A_2 U_{1Y}^{(1)} + \frac{\mathrm{i}U_1^{(1)}U_2^{(1)}(\hat{s}_1-\hat{s}_2)\sin(\theta_1-\theta_2)}{\hat{s}_1\hat{s}_2},
\end{aligned} \tag{14}$$

with solutions

$$\begin{aligned}
V_A^{(2)} &= \int_{-\infty}^{X}\int_{-\infty}^{X_1} K_A^{(2a)} A_1(X_0)A_2(X_1)\,\mathrm{e}^{-\mathrm{i}\widetilde{Y}_1(X-X_0)-\mathrm{i}\widetilde{Y}_2(X-X_1)}\,\mathrm{d}X_0\,\mathrm{d}X_1\\
&\quad + \int_{-\infty}^{X}\int_{-\infty}^{X_1} K_A^{(2b)} A_1(X_1)A_2(X_0)\,\mathrm{e}^{-\mathrm{i}\widetilde{Y}_1(X-X_1)-\mathrm{i}\widetilde{Y}_2(X-X_0)}\,\mathrm{d}X_0\,\mathrm{d}X_1,\\
U_A^{(2)} &= \int_{-\infty}^{X}\int_{-\infty}^{X_1} K_A^{(2c)} A_1(X_0)A_2(X_1)\,\mathrm{e}^{-\mathrm{i}\widetilde{Y}_1(X-X_0)-\mathrm{i}\widetilde{Y}_2(X-X_1)}\,\mathrm{d}X_0\,\mathrm{d}X_1\\
&\quad + \int_{-\infty}^{X}\int_{-\infty}^{X_1} K_A^{(2d)} A_1(X_1)A_2(X_0)\,\mathrm{e}^{-\mathrm{i}\widetilde{Y}_1(X-X_1)-\mathrm{i}\widetilde{Y}_2(X-X_0)}\,\mathrm{d}X_0\,\mathrm{d}X_1,\\
W_A^{(2)} &= \frac{\mathrm{i}V_{AY}^{(2)} - U_A^{(2)}(\hat{c}_1+\hat{c}_2)}{\hat{s}_1+\hat{s}_2},
\end{aligned} \tag{15}$$

where the kernels $K_A^{(2)}$ are given in the appendix. The remaining interaction terms we require can be recovered from this term using the following transformations

$$\begin{aligned}
\begin{pmatrix} U_B^{(2)} \\ V_B^{(2)} \end{pmatrix} &= (\hat{s}_2 \to -\hat{s}_2)\begin{pmatrix} U_A^{(2)} \\ V_A^{(2)} \end{pmatrix},\\
\begin{pmatrix} U_C^{(2)} \\ V_C^{(2)} \end{pmatrix} &= \begin{pmatrix} \hat{c}_2 \to -\hat{c}_2 \\ A_2 \to A_2^* \\ U_2 \to U_2^* \end{pmatrix}\begin{pmatrix} U_A^{(2)} \\ V_A^{(2)} \end{pmatrix},\\
\begin{pmatrix} U_D^{(2)} \\ V_D^{(2)} \end{pmatrix} &= (\hat{s}_2 \to -\hat{s}_2)\begin{pmatrix} U_C^{(2)} \\ V_C^{(2)} \end{pmatrix}.
\end{aligned} \tag{16}$$

We should mention that special attention must be paid several of these nonlinear terms, specifically the cross-flow terms proportional to $\mathrm{e}^{2\mathrm{i}\hat{s}_1 z}$ and the terms proportional to $\mathrm{e}^{\mathrm{i}(\hat{c}_1\xi_1-\hat{c}_2\xi_2+(\hat{s}_1\pm\hat{s}_2)z)}$, because there are jumps in these terms across the critical layer which require that additional terms be added to the outer expansion in order that the two flows match together. The reason that this occurs with the $\mathrm{e}^{\mathrm{i}(\hat{c}_1\xi_1-\hat{c}_2\xi_2+(\hat{s}_1\pm\hat{s}_2)z)}$ terms but not the $\mathrm{e}^{\mathrm{i}(\hat{c}_1\xi_1+\hat{c}_2\xi_2+(\hat{s}_1\pm\hat{s}_2)z)}$ ones is that a necessary condition for such a jump to occur is that the coefficient of Y in the exponentials in these terms must vanish somewhere in the range of integration. The jumps in the $\mathrm{e}^{\mathrm{i}(\hat{c}_1\xi_1-\hat{c}_2\xi_2+(\hat{s}_1\pm\hat{s}_2)z)}$ terms are very similar to the jumps in the cross-flow terms, and so we are able to handle these two sets of jumps in a similar fashion, using the approach taken by Goldstein and Choi [1]. Goldstein and Choi added additional terms to their outer expansion to generate jumps there that could be matched to these new jumps in the inner expansion, specifically they added a spanwise mean flow component in the form of a streamwise vortex motion of the form

$$2\big(\varepsilon u_{02}(y,X),\varepsilon^{4/3}v_{02}(y,X),\varepsilon^{5/3}w_{02}(y,X)\big)\cos 2\hat{s}_1 z, \tag{17}$$

where for the tanh y mixing layer [9]

$$u_{02} = \pm 2\hat{s}_1 C_{02}^{(1)\pm}\,\mathrm{e}^{\mp 2\hat{s}_1 y}, \tag{18}$$

where $\pm$ denotes above and below the critical layer respectively. This additional term has been included in virtually every subsequent nonlinear critical layer study on three-dimensional transition, either explicitly or implicitly, including [3–12,15]. Wu and Stewart [18] used a similar approach to deal with the difference term in their phase-locked interaction, and we will similarly add an additional term

$$\varepsilon^{4/3}\,\mathrm{e}^{\mathrm{i}(\hat{c}_1\xi_1-\hat{c}_2\xi_2+\hat{s}_1 z)}\big[(u_C,v_C,w_C)\,\mathrm{e}^{\mathrm{i}\hat{s}_2 z}+(u_D,v_D,w_D)\,\mathrm{e}^{-\mathrm{i}\hat{s}_2 z}\big] \tag{19}$$

to the outer expansion, which ensures that we can match the inner and outer expansions together; in this additional term, which represents a vortex motion, the velocity components in the outer expansion are given by

$$\begin{aligned}
v_C &= C_C\left[1+\frac{\tanh|y|}{\sqrt{(\hat{c}_1-\hat{c}_2)^2+(\hat{s}_1+\hat{s}_2)^2}}\right]\mathrm{e}^{-\sqrt{(\hat{c}_1-\hat{c}_2)^2+(\hat{s}_1+\hat{s}_2)^2}|y|},\\
u_C &= \frac{\mathrm{i}}{(\hat{c}_1-\hat{c}_2)^2+(\hat{s}_1+\hat{s}_2)^2}\left[(\hat{c}_1-\hat{c}_2)v_{Cy}+\frac{2(\hat{s}_1+\hat{s}_2)^2 v_C}{(\hat{c}_1-\hat{c}_2)\sinh 2y}\right],\\
w_C &= \frac{i(\hat{s}_1+\hat{s}_2)}{(\hat{c}_1-\hat{c}_2)^2+(\hat{s}_1+\hat{s}_2)^2}[v_{Cy}-2v_C\,\mathrm{csch}\,2y],
\end{aligned} \tag{20}$$

with the corresponding pressure term given by

$$p_C = \frac{\mathrm{i}(\hat{c}_1-\hat{c}_2)}{(\hat{c}_1-\hat{c}_2)^2+(\hat{s}_1+\hat{s}_2)^2}[v_C\,\mathrm{sech}\,y - v_{Cy}\tanh y]. \tag{21}$$

Terms equivalent to these can be found in §3 of [18]. The other additional velocity terms can be recovered from (20) by making the transformation $\hat{s}_2\to-\hat{s}_2$. It follows that for example there is a jump in v_{Cy} across the critical layer given by

$$[v_{C,y}]_{0-}^{0+} = -\frac{2C_C[1+(\hat{c}_1-\hat{c}_2)^2+(\hat{s}_1+\hat{s}_2)^2]}{\sqrt{(\hat{c}_1-\hat{c}_2)^2+(\hat{s}_1+\hat{s}_2)^2}}, \tag{22}$$

and this jump can match to the jump in $V_{CY}^{(2)}$ across the inner region. It follows that the presence of these additional terms (17), (20) in the outer expansion allows us to match the inner and outer expansion together.

At the next order, in the inner expansion we must calculate the jump in the fundamental, which will enable us to write our amplitude equations. To simplify the algebra, we write

$$V_1^{(3)} = V_1^{(3a)} + V_1^{(3b)}, \tag{23}$$

where $V_1^{(3b)}$ is given in the appendix and consists of terms that do not contribute to the jump while $V_1^{(3a)}$ obeys an equation of the form

$$\left[\frac{\partial}{\partial X}+\mathrm{i}\widetilde{Y}_1\right]V_{1YY}^{(3a)} = F_1^{(3a)}, \tag{24}$$

where the forcing term $F_1^{(3a)}$ and $V_{1YY}^{(3a)}$ itself are given in the appendix. We must now calculate the jump across the critical layer. Matching to the outer solution, we find that the jump must satisfy

$$\int_{-\infty}^{\infty}\big(V_{1YY}^{(3)}+A_1\big)\,\mathrm{d}Y = 2\big[D_1^+-D_1^-\big], \tag{25}$$

but we already know from the outer expansion (7) that

$$D_1^+-D_1^- = 2\mathrm{i}A'\cos\theta, \tag{26}$$

and combining the outer (7) and inner (25) jumps leads to our first amplitude equation,

$$
\begin{aligned}
&\mathrm{i}\gamma_1 A_1 + \left(\frac{2\mathrm{i}\hat{c}_1}{\pi} - \frac{1}{\hat{c}_1}\right)A_1' = 2\hat{c}_1^3\hat{s}_1^2\cos 2\theta_1 \int_0^\infty\int_0^\infty \tau_0\big[2\hat{s}_1^2\tau_1(\tau_0+\tau_1) + \tau_0(2\tau_0+\tau_1)\big] \\
&\quad\times A_1(X-\tau_0)A_1(X-\tau_0-\tau_1)A_1^*(X-2\tau_0-\tau_1)\,\mathrm{d}\tau_0\,\mathrm{d}\tau_1 \\
&\quad+ 2\hat{c}_2\hat{s}_2^2\hat{c}_1^2\cos 2\theta_1 \int_0^\infty\int_0^\infty \mathrm{e}^{\mathrm{i}\tilde{c}_1\tau_0(\gamma_1-\gamma_2)}\tau_0^3 A_1(X-\tau_0)A_2(X-\tau_0-\tau_1)A_2^*\left(X-\tau_1-\tau_0\left(1+\frac{\hat{c}_1}{\hat{c}_2}\right)\right)\mathrm{d}\tau_0\,\mathrm{d}\tau_1 \\
&\quad+ \frac{2\hat{s}_1^2\hat{c}_1^2\hat{c}_2^2 H(\theta_2-\theta_1)}{(\hat{c}_2-\hat{c}_1)^3}\int_0^\infty\int_0^\infty \mathrm{e}^{\mathrm{i}\tilde{c}_1(\tau_1+\tau_0)(\gamma_1-\gamma_2)}\tau_1(\tau_0+\tau_1) \\
&\quad\times\big[\tau_1\hat{s}_2^2\big(4\hat{c}_1^2\hat{c}_2^2-\hat{c}_2^2-\hat{c}_1^2\big) - \tau_0\hat{c}_1\big(\hat{c}_1^2\cos 3\theta_2 - \hat{c}_1\cos 2\theta_2 + \hat{c}_2\hat{s}_2^2\big)\big] \\
&\quad\times A_2(X-\tau_0)A_2^*\left(X-\frac{\hat{c}_2\tau_1+\hat{c}_1\tau_0}{\hat{c}_2-\hat{c}_1}\right)A_1\left(X-\frac{\hat{c}_2(\tau_1+\tau_0)}{\hat{c}_2-\hat{c}_1}\right)\mathrm{d}\tau_0\,\mathrm{d}\tau_1 \\
&\quad+ 2\hat{c}_1^2\hat{s}_2^2 H(\theta_1-\theta_2)\int_0^\infty\int_0^\infty \mathrm{e}^{\mathrm{i}\tilde{c}_1(\tau_1+\tau_0)(\gamma_1-\gamma_2)}\tau_0(\tau_0+\tau_1)\big[(\tau_1+\tau_0)\hat{c}_2\cos 2\theta_1 - \big(\cos 3\theta_1\hat{c}_2^2+\hat{c}_1\hat{s}_1^2\big)\tau_1\big] \\
&\quad\times A_2(X-\tau_0)A_2^*\left(X-\tau_1-\frac{(\tau_1+\tau_0)\hat{c}_1}{\hat{c}_2}\right)A_1(X-\tau_1-\tau_0)\,\mathrm{d}\tau_0\,\mathrm{d}\tau_1 \\
&\quad+ \frac{H(\theta_2-\theta_1)\sin^2 2\theta_2}{2\hat{c}_1^2}\int_0^\infty\int_0^\infty \mathrm{e}^{\mathrm{i}\tilde{c}_2\tau_1(\gamma_2-\gamma_1)}(\tau_1+\tau_0)\big(\hat{c}_2(\tau_0+\tau_1)-\hat{c}_1\tau_1\big) \\
&\quad\times\big[\tau_1\hat{c}_1^2\hat{s}_1^2\cos 2\theta_2 + \cos 2\theta_1\hat{c}_2^2\big(\hat{s}_1^2\tau_1+\tau_2\big)\big]A_2\left(X+\tau_1-\frac{\hat{c}_2(\tau_0+\tau_1)}{\hat{c}_1}\right) \\
&\quad\times A_2^*\left(X-\tau_0-\frac{\hat{c}_2(\tau_0+\tau_1)}{\hat{c}_1}\right)A_1\left(X-\frac{\hat{c}_2(\tau_0+\tau_1)}{\hat{c}_1}\right)\mathrm{d}\tau_0\,\mathrm{d}\tau_1.
\end{aligned}
\tag{27}
$$

The second amplitude equation can be written down using symmetry, simply interchanging the indices "1" and "2" in the first equation. There are several points to notice about these equations. Firstly, the terms on the left-hand side are linear terms. If the amplitudes A_1 and A_2 were both small, all of the terms on the right-hand side would disappear and we would be left with linear equations, with solutions $A_1 = A_{10}\,\mathrm{e}^{\sigma_1 X}$, where the linear growth rate

$$
\sigma_1 = \frac{\pi\gamma_1\hat{c}_1}{2\hat{c}_1^2+\mathrm{i}\pi},
\tag{28}
$$

with a similar expression for A_2. Turning to the nonlinear terms, the first term on the right-hand side of the equation is of course the familiar Goldstein and Choi [1] expression for the cubic self-interaction of a pair of waves with itself, and if only one pair were present, our equation would, as expected, reduce to that of Goldstein and Choi. The remaining terms represent interactions *between* the pairs of waves, and are new to the present analysis. These terms mean that during the nonlinear stage, each pair of waves affects the other. The presence of the Heaviside step functions, $H(\theta_2-\theta_1)$ and $H(\theta_1-\theta_2)$, in the equation above should also be noted, meaning that some terms will only appear in the equation for the wave with the larger θ while others will only appear in that for the smaller θ, meaning of course that the two equations are in reality far less symmetric than they appear to be at first sight.

We should also address what happens when one pair of waves is much larger than the other. If we suppose that $|A_2| \sim \mathrm{O}(1)$ but $|A_1| \ll 1$, so that $|A_1| \ll |A_2|$, then in the above equations, we can neglect terms that are either quadratic or cubic in A_1; under those conditions, the equation for A_2 will reduce to that of Goldstein and Choi,

$$
\begin{aligned}
\mathrm{i}\gamma_2 A_2 + \left(\frac{2\mathrm{i}\hat{c}_2}{\pi} - \frac{1}{\hat{c}_2}\right)A_2' &= 2\hat{c}_2^3\hat{s}_2^2\cos 2\theta_2 \int_0^\infty\int_0^\infty \tau_0\big[2\hat{s}_2^2\tau_1(\tau_0+\tau_1) + \tau_0(2\tau_0+\tau_1)\big] \\
&\quad\times A_2(X-\tau_0)A_2(X-\tau_0-\tau_1)A_2^*(X-2\tau_0-\tau_1)\,\mathrm{d}\tau_0\,\mathrm{d}\tau_1.
\end{aligned}
\tag{29}
$$

In the equation for A_1, the first (self-interaction) term on the right-hand side of (31) will disappear, but the remaining nonlinear terms (representing interactions between the pairs) will remain. Thus, the larger pair affects the smaller, but not vice versa. This can be seen more formally by replacing A_1 by νA_1 (with $\nu \ll 1$) in the amplitude equations and neglecting higher order terms in ν. One effect of this is obvious and important: when A_2 undergoes the finite-time singularity found by Goldstein and Choi [1], A_1 will also undergo explosive growth. This last result is significant: suppose we have a situation where a single pair of waves is forced, and undergoes the Goldstein and Choi mechanism. If there is background noise in the experiment, there will be other oblique waves with smaller amplitudes in that background noise, and when the forced wave experiences explosive growth, those other waves in the background will also experience very rapid growth, and because of this, one limit of the equations present here (27) can be thought of as a secondary instability of a pair of oblique waves.

As with similar equations of this form, the solutions A_1 and A_2 will become singular at some finite distance downstream, with the structure of the singularity being the same as that found in [1], namely

$$A_1 \sim b_1/(X_s - X)^{5/2+\mathrm{i}\psi_1}, \qquad A_2 \sim b_2/(X_s - X)^{5/2+\mathrm{i}\psi_2}, \tag{30}$$

where ψ_1 and ψ_2 are real (so the real parts of the exponent are the same for the two modes but the imaginary parts will differ) and b_1 and b_2 are complex. This singularity would be manifested in a real flow as extremely rapid growth, marking the onset of a subsequent more nonlinear stage governed by the complete Euler equations.

Although this paper has been concerned with interactions between two pairs of oblique waves superimposed on a mixing layer, it would be fairly straightforward to extend it to include a third pair, with amplitude A_3 at an angle θ_3, or possibly an even greater number of pairs: provided that the phase velocities of these waves were all (roughly) equal, there would be an interaction between them. If a third pair were to be included, the amplitude equations for A_1 and A_2 would need to be modified, so that for example (27) would contain an additional group of nonlinear terms coming from the interaction between the pair of waves at angle θ_1 and that at θ_3; these new terms would be of the same form as the terms coming from the interaction between θ_1 and θ_2 modes, but with the subscript "2" replaced by the subscript "3".

It is also possible to deduce from our amplitude equations what would happen if the disturbance consisted not of two pairs or oblique waves but rather one pair of waves at an angle θ_1 together with a single wave at an angle θ_2. In such a case, the amplitude equation for the single wave at θ_2 would still contain the nonlinear terms coming from the interaction with the pair at θ_1 but would lack the self-interaction terms of the Goldstein–Choi equation, while the amplitude equation for the pair at θ_1 would still contain the Goldstein–Choi self-interaction terms along with some but not all of the other nonlinear terms, with the terms coming from $V_A^{(2)}$ and $V_D^{(2)}$ remaining but those from $V_B^{(2)}$ and $V_C^{(2)}$ absent. Because of this, intuitively one would expect the nonlinear interaction between a pair of oblique waves and a single oblique wave to be weaker than that between two pairs.

Finally in this section, we make some remarks on the effects of viscosity. Although weak viscosity could be added to the present analysis by writing $\mathrm{Re}^{-1} = \varepsilon\lambda$, where λ is the Benney–Bergeron parameter, a full study of the effects of viscosity is beyond the scope of this article. However, we can make some inferences based on other studies that have included viscous effects, namely those of Wu [6] and Lee [15]. Wu's results indicate that viscosity delays the occurrence of the finite distance singularity but does not appear to be able to eliminate it. Because of this, we are able to infer that our result is still meaningful at high Reynolds numbers.

4. Numerical solution of the amplitude equations

In this section, we present some sample numerical solutions of the coupled evolution equations presented in the preceding section for the amplitudes A_1 and A_2. Eq. (27), together with its counterpart for the other pair of waves, was solved numerically using a numerical method due to [1]. The specific code used was a modified version of that used in [12]. The numerical method involved two main components at each time step. Firstly, the integrals in (27) were evaluated numerically by truncating the infinite domain to a finite interval, on which the integrals were evaluated using a Newton–Coates formula, while the infinite tails were estimated analytically using the known asymptotic behavior of the amplitudes as $X \to -\infty$. In the second stages, once the integrals had been evaluated, we were able to calculate A_1' and A_2' and thereby advance the amplitudes A_1 and A_2 using a tenth-order Runge–Kutta scheme with a step size of 2×10^{-3}. The integration was started at some initial point X_0 during the linear growth stage, when the amplitudes were sufficiently small that the nonlinear terms in (27) could be neglected and we were able to assume that we had linear growth for $X < X_0$ with $A_1 = A_{10}\,\mathrm{e}^{\sigma_1 X}$ and $A_2 = A_{20}\,\mathrm{e}^{\sigma_1 X}$. For the runs presented here, we took $X_0 = -100$.

Following a similar strategy to that used by Wu [18], to make the integration easier, we chose the angles of the waves so that $\hat{c}_1/\hat{c}_2 = 2$, which meant that during the integration we only needed the amplitudes at regularly spaced grid points; for more general angles, some sort of interpolation would be required. The specific angles used in our calculations were $\theta_1 = 60°$, so that $\hat{c}_1 = 1/2$, and $\theta_2 = \arccos(1/4) \approx 75.52$. Four runs are presented here, and in each of these, we took the initial amplitudes A_{10} and A_{20} to be real. In each of Run 1 and Run 2, only one mode was present, so we were solving the Goldstein–Choi equation equation. In Run 1, we set $A_{10} = 10^{-2}$, while in Run 2, we set $A_{20} = 10^{-5}$. These two initial runs were partially to validate the code against earlier results and partially to give a benchmark against which to compare the later runs with both modes present. Two subsequent runs, Run 3 and Run 4, were performed with both modes present. These two runs each took roughly one month

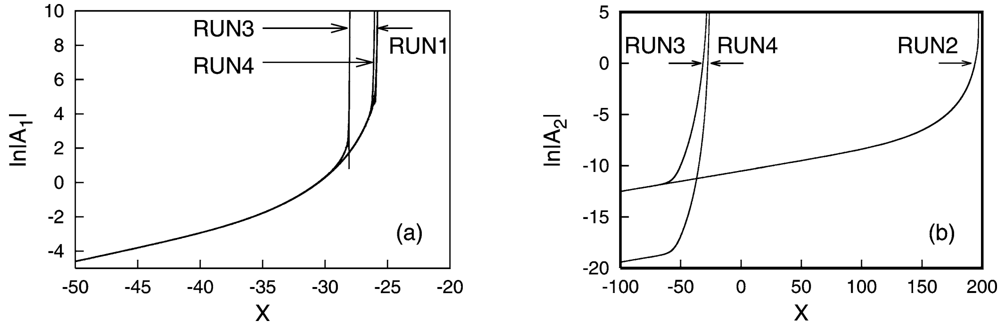

Fig. 1. (a) evolution of $|A_1|$ in Runs 1, 3 and 4; (b) evolution of $|A_2|$ in Runs 2, 3 and 4.

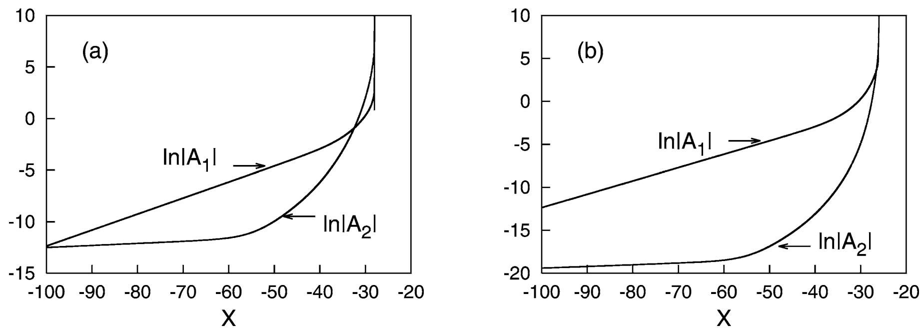

Fig. 2. Evolution of $|A_1|$ and $|A_2|$: (a) Run 3; (b) Run 4.

of CPU time on a Itanium 2 chip running at 1 Gigahertz. The computations were performed on the Sharcnet Canada cluster. In both Run 3 and Run 4, A_{10} took the same value as in Run 1. In Run 3, A_{20} took the same initial value as in Run 2, while in Run 4, A_{20} was set to the much smaller value of 10^{-8}.

Our numerical results are presented graphically in Figs. 1 and 2. In Fig. 1(a), we present the evolution of $\ln|A_1|$ for Runs 1, 3 and 4. If there was no interaction between the modes, all three lines would be identical since they have the same initial condition. However, it can be seen that while the linear stage is the same for all 3 runs, the location of the blow-up is different for each of the three runs, occurring earliest for Run 3, then for Run 4 and finally for Run 1. In addition, while the behavior of Run 1 appears to be monotonic during the blow-up process, there is a weak oscillation present in Run 4 and a strong oscillation present in Run 3. In the cases studied, this oscillation appears to occur only in $|A_1|$ not $|A_2|$.

In Fig. 1(b), we present the evolution of $\ln|A_2|$ for Runs 2, 3 and 4. Initially, Runs 2 and 4 are identical, since they had the same initial condition. However, the blow-up occurs substantially earlier for Runs 3 and 4 than it does for Run 2. The blow-up appears to start at roughly the same time for Runs 3 and 4, and presumably is caused by the amplitude of the other wave $|A_2|$ reaching a large enough value, but the blow-up process takes much longer in Run 4 than in Run 3, presumably due to the smaller value of $|A_2|$ in Run 4.

The evolution of both amplitudes for Run 3 is presented in Fig. 2(a), with a similar plot in Fig. 2(b) for Run 4. In each of these runs, the evolution of A_2 becomes nonlinear first, due to interactions with A_1, and this nonlinearity eventually couples back to A_1 when $|A_2|$ has grown sufficiently large.

Although we have only presented two runs with both pairs present, it is clear from those runs that an interaction between the pairs can take place, and that the appearance of the finite time singularity can be accelerated by that interaction.

5. Concluding remarks

In the preceding sections, coupled amplitude equations consisting of (27) and its counterpart for the other pair of waves were derived governing the interaction of two pairs of oblique waves superimposed upon a mixing layer. The two pairs were assumed to have the same wavenumber and their amplitudes were nominally of the same order, but their phase velocities were slightly different and they were inclined at *different* angles to the plane of the mixing layer. Because the phase velocities of the waves were almost equal, the interactions can be thought of as an example of the phase-locked interaction of [18]. As in [1], these equations are of integro-differential form with cubic nonlinearity and depend upon the entire history of the flow.

An unusual feature of these equations however is that the arguments of A_1 and A_2 in the integrals in (27) involve the angles at which the waves are inclined to the shear layer; this happened in [18] but not in [1]. As is normal for nonlinear critical layer analyses involving pairs of oblique waves, these equations describe two successive stages of the evolution process: the initial linear growth stage for small disturbances, when the nonlinear terms in the evolution equations vanish, and a fully nonlinear stage. It can be seen from the equations that during this nonlinear stage, the pairs affect each other via the cubic nonlinearities in the amplitude equations. In addition, there is a third stage in which the waves experience a singularity at a finite distance downstream.

In Section 4, we presented some sample numerical solutions of the coupled amplitude equations to illustrate the points mentioned above. These numerical solutions, which were presented in Figs. 1 and 2, confirmed that nonlinear interactions between the pairs can indeed take place, and also demonstrated that the appearance of the finite time singularity could be accelerated by these nonlinear interactions.

It is also interesting to note that if one of the pairs were much smaller than the other, then the equation for the dominant pair is simply that given by Goldstein and Choi [1], which is to be expected since the present study is an extension of [1], while the evolution of the smaller pair is influenced by the presence of the larger pair. One particular scenario that could be envisaged therefore only one pair is forced but another (much smaller) pair is present in the background noise. When the nonlinear stage presented here is reached, the forced pair would experience very rapid growth and the singularity after a finite distance, and this singularity would be coupled back to the background pair which would also experience very rapid growth. In reality, in an experiment, there might be many such pairs of waves in the background noise and each pair would undergo this mechanism, and in Section 3 we discussed how the amplitude equations could be extended to include three or more pairs of waves.

Finally, as we mentioned in Section 1, we are unaware of either any experiments or any numerical simulations which address the situation covered here, so that comparisons of our theory with experiments is not possible, and we would suggest that such experiments and/or simulations might be a worthwhile endeavor.

Appendix. Details of the analysis

The self-interaction terms in Section 3 are given by

$$U_{20}^{(2)} = \frac{\mathrm{i}V_{20Y}^{(2)}}{2\hat{c}_1},$$

$$V_{20}^{(2)} = -4\mathrm{i}\hat{s}_1^2\hat{c}_1 \int_{-\infty}^{X}\int_{-\infty}^{X_1} \left[\frac{4\hat{s}_1^2(X-X_0)(X-X_1)+(X_0-X_1)^2}{(2X-X_1-X_0)^2}\right] A_1(X_1)A_1(X_0)\,\mathrm{e}^{\mathrm{i}\widetilde{Y}_1(X_0+X_1-2X)}\,\mathrm{d}X_0\,\mathrm{d}X_1,$$

$$U_{22}^{(2)} = \mathrm{i}\hat{s}_1^2\hat{c}_1 \int_{-\infty}^{X}\int_{-\infty}^{X_1} (X_0-X_1)A_1(X_1)A_1(X_0)\,\mathrm{e}^{\mathrm{i}\widetilde{Y}_1(X_0+X_1-2X)}\,\mathrm{d}X_0\,\mathrm{d}X_1,$$

$$W_{22}^{(2)} = \frac{-\hat{c}_1 U_{22}^{(2)}}{\hat{s}_1},$$

$$U_{02}^{(2)} = \mathrm{i}\hat{s}_1^2\hat{c}_1 \int_{-\infty}^{X}\int_{-\infty}^{X_1} \left[\frac{4\hat{s}_1^2(X-X_1)^2}{X_0-X_1}+X_0+X_1-2X\right]\left[A_1(X_0)A_1^*(X_1)\,\mathrm{e}^{\mathrm{i}\widetilde{Y}_1(X_0-X_1)}-c.c.\right]\mathrm{d}X_0\,\mathrm{d}X_1,$$

$$V_{02}^{(2)} = 2\mathrm{i}\hat{s}_1^2\hat{c}_1 \int_{-\infty}^{X}\int_{-\infty}^{X_1} \left[1-\frac{4\hat{s}_1^2(X-X_1)}{X_0-X_1}\right]\left[A_1(X_0)A_1^*(X_1)\,\mathrm{e}^{\mathrm{i}\widetilde{Y}_1(X_0-X_1)}-c.c.\right]\mathrm{d}X_0\,\mathrm{d}X_1,$$

$$W_{02}^{(2)} = \frac{\mathrm{i}\hat{c}_1 V_{02Y}^{(2)}}{2\hat{s}_1},$$

$$U_{00}^{(2)} = 2\mathrm{i} \int_{-\infty}^{X}\int_{-\infty}^{X_1} (X_1-X_0)\left[\hat{s}_1^2\hat{c}_1 A_1(X_1)A_1^*(X_0)\,\mathrm{e}^{\mathrm{i}\widetilde{Y}_1(X_1-X_0)}+\hat{s}_2^2\hat{c}_2 A_2(X_1)A_2^*(X_0)\,\mathrm{e}^{\mathrm{i}\widetilde{Y}_2(X_1-X_0)}-c.c.\right]\mathrm{d}X_0\,\mathrm{d}X_1.$$

The kernels for Section 3 are given by

$$K_A^{(2a)} = -\frac{\mathrm{i}\hat{s}_1 \sin(\theta_1-\theta_2) L_A^{(2a)}}{[(X-X_0)\hat{c}_1 + (X-X_1)\hat{c}_2]^2},$$
$$L_A^{(2a)} = \hat{c}_2\hat{c}_1\cos(\theta_1-\theta_2)X_1(X_1-2X_0) + \hat{c}_1^2X_0^2 + \hat{s}_2\big[\hat{c}_1X(2X_0-X) - \hat{c}_2\sin(\theta_1-\theta_2)(X-X_1)^2\big],$$
$$K_A^{(2b)} = -\frac{\mathrm{i}\hat{s}_2 \sin(\theta_1-\theta_2) L_A^{(2b)}}{[(X-X_1)\hat{c}_1 + (X-X_0)\hat{c}_2]^2},$$
$$L_A^{(2b)} = \hat{c}_1\hat{c}_2\cos(\theta_1-\theta_2)X_1(X_1-2X_0) + \hat{c}_2^2X_0^2 + \hat{s}_1\sin(\theta_1-\theta_2)\big[\hat{c}_1(X_1-X)^2 + \hat{c}_2X(X-2X_0)\big],$$
$$K_A^{(2c)} = \frac{\mathrm{i}\hat{s}_1 L_A^{(2c)}}{[(X-X_0)\hat{c}_1 + (X-X_1)\hat{c}_2]^2},$$
$$\begin{aligned}
L_A^{(2c)} = {}& \hat{s}_1\hat{c}_1^3X_0\big(X_0^2 - 2XX_0 + 3X^2\big) + \hat{c}_1\hat{c}_2\big(\hat{c}_2\hat{s}_1^3 + \hat{c}_1^3\hat{s}_2\big)X^2X_1 + \hat{c}_1^2\hat{s}_1\hat{c}_2\big[X_0^2(X+X_1) + 3X^2X_0 + X_1^2(X-X_0)\big] \\
& - \hat{c}_1^2\hat{s}_2\hat{c}_2\big(3X^2X_0 + 3X^2X_1 + X_1^2X_0\big) - \hat{c}_1\hat{s}_2\hat{c}_2^2\big(X^2X_0 + X_1^2X_0 + 5X^2X_1\big) \\
& + \big[\hat{c}_1^2\hat{c}_2\hat{s}_2^3\big(1 + \hat{c}_1\hat{c}_1 + 2\hat{c}_2^2\big) + \hat{s}_1\hat{s}_2\hat{c}_1\hat{c}_2^2(\hat{s}_1 + \hat{s}_2)\big]X_1^2X \\
& + \sin(\theta_1-\theta_2)\big[\hat{c}_1^2\hat{s}_1\hat{s}_2X\big(X^2 + X_0^2\big) - \hat{c}_1\hat{c}_2\hat{s}_2^2X\big(X^2 + 4X_0X_1\big) \\
& + \hat{s}_1\hat{c}_1\hat{s}_2\hat{c}_2X\big(X^2 + 2X_0X_1\big) + \hat{s}_2^2\hat{c}_2^2\big(X_1^3 - X^3 - 3X_1^2X\big) \\
& + 2\hat{c}_1^3\hat{c}_2XX_0(X_1 - X) - \hat{c}_1^2\hat{c}_2^2X^2(2X_0 + X_1) - \hat{c}_1\hat{c}_2^3X^2X_1\big] \\
& + \cos(\theta_1-\theta_2)\big[\hat{c}_1\hat{s}_2\hat{c}_2^2\big(2X_0X^2 + 2X_0X_1^2 - X_1^3\big) + \hat{c}_1^2\hat{c}_2\hat{s}_2X_0^2(X_1 - X) - \hat{c}_1^2\hat{s}_1\hat{c}_2X_1X_0^2 + 3\hat{s}_2\hat{c}_2^3X^2X_1\big],
\end{aligned}$$
$$K_A^{(2d)} = \frac{\mathrm{i}\hat{s}_2 L_A^{(2d)}}{[(X-X_1)\hat{c}_1 + (X-X_0)\hat{c}_2]^2},$$
$$\begin{aligned}
L_A^{(2d)} = {}& \hat{s}_2\hat{c}_2^3X_0\big(X_0^2 - 2XX_0^3X^2\big) + \hat{c}_1\hat{c}_2^2\hat{s}_2\big(2X_1X_0^2 - X_1^2X_0 + X_1^2X - XX_0^2 + 3X^2X_0\big) \\
& + \hat{c}_1^2\hat{s}_1\hat{c}_2\big[X_1^2(4X - X_0) - X^2(X_0 + 5X_1)\big] + \hat{c}_1\hat{s}_1\hat{c}_2^2\big[X_1^2(X - X_0) - 3X^2X_0\big] + \big(\hat{c}_1^2\hat{c}_2\hat{s}_2 + 3\hat{c}_1^3\hat{s}_1\big)X^2X_1 \\
& + \sin(\theta_1-\theta_2)\big[\hat{c}_1^2\hat{s}_1^2\big(X^3 - X_1^3\big) + \hat{c}_2\hat{c}_1^3X^2X_1 - \hat{c}_1\hat{s}_1\hat{s}_2\hat{c}_2\big(X^3 + 2X_0X_1\big) \\
& + \hat{c}_1\hat{s}_1^2\hat{c}_2\big(X^3 + 4X_0X_1\big) + \hat{c}_1\hat{c}_2^3\big(2X^2X_0 - 2X_0X_1 + X^2X_1\big) \\
& - \hat{s}_1\hat{s}_2\hat{c}_2^2X\big(X^2 + X_0^2\big) + \hat{c}_1^2\hat{c}_2^2\big(X_1X_0^2 + 2X^2X_0 + X^2X_1\big)\big] \\
& + \hat{c}_1\cos(\theta_1-\theta_2)\big[2\hat{c}_1\hat{s}_1\hat{c}_2X_0\big(X_1^2\big) + \hat{c}_1\hat{s}_1\hat{c}_2\big(X_1^2X - X_1^3\big) - \hat{c}_2^2X_0^2(\hat{s}_2X_1 + \hat{s}_1X) + 3\hat{c}_1^2\hat{s}_1X_1^2X(X - X_1)\big].
\end{aligned}$$

The velocity term omitted in (23) in Section 3 is

$$\begin{aligned}
V_1^{(3b)} = {}& \frac{\hat{s}_1\sin(\theta_1-\theta_2)}{\hat{c}_1\hat{s}_2}\left[\frac{U_2^{(1)}V_{CY}^{(2)}}{\hat{s}_1 - \hat{s}_2} - \frac{U_2^{(1)*}V_{AY}^{(2)}}{\hat{s}_1 + \hat{s}_2}\right] + \frac{\hat{s}_1\sin(\theta_1+\theta_2)}{\hat{c}_1\hat{s}_2}\left[\frac{U_2^{(1)*}V_{BY}^{(2)}}{\hat{s}_1 - \hat{s}_2} + \frac{U_2^{(1)}V_{CY}^{(2)}}{\hat{s}_1 + \hat{s}_2}\right] \\
& + \frac{\mathrm{i}\sin^2(\theta_1+\theta_2)}{\hat{c}_1\hat{s}_2}\left[\frac{U_2^{(1)*}U_B^{(2)}}{\hat{s}_1 - \hat{s}_2} - \frac{U_2^{(1)}U_C^{(2)}}{\hat{s}_1 + \hat{s}_2}\right] + \frac{\mathrm{i}\sin^2(\theta_1-\theta_2)}{\hat{c}_1\hat{s}_2}\left[\frac{U_2^{(1)}U_D^{(2)}}{\hat{s}_1 - \hat{s}_2} - \frac{U_2^{(1)*}U_A^{(2)}}{\hat{s}_1 + \hat{s}_2}\right] \\
& + U_1^{(1)}\big(V_{02Y}^{(2)} - 2\mathrm{i}\hat{c}_1U_{02}^{(2)}\big) + U_1^{(1)*}V_{20Y}^{(2)}.
\end{aligned}$$

The forcing term in (24) is

$$\begin{aligned}
F_1^{(3a)} = {}& \frac{\hat{s}_1\hat{c}_2\cos(\theta_1-\theta_2)}{\hat{c}_1}\partial_{YYY}^3\left[\frac{A_2V_D^{(2)}}{\hat{s}_2 - \hat{s}_1} - \frac{A_2^*V_A^{(2)}}{\hat{s}_1 + \hat{s}_2}\right] - \frac{\hat{s}_1\hat{c}_2\cos(\theta_1+\theta_2)}{\hat{c}_1}\partial_{YYY}^3\left[\frac{A_2V_C^{(2)}}{\hat{s}_1 + \hat{s}_2} + \frac{A_2^*V_B^{(2)}}{\hat{s}_1 - \hat{s}_2}\right] \\
& + \frac{\mathrm{i}\hat{c}_2\sin 2(\theta_2-\theta_1)}{2\hat{c}_1}\partial_{YY}^2\left[\frac{A_2^*U_A^{(2)}}{\hat{s}_1 + \hat{s}_2} + \frac{A_2U_D^{(2)}}{\hat{s}_2 - \hat{s}_1}\right] + \frac{\mathrm{i}\hat{c}_2\sin 2(\theta_1+\theta_2)}{2\hat{c}_1}\partial_{YY}^2\left[\frac{A_2U_C^{(2)}}{\hat{s}_1 + \hat{s}_2} + \frac{A_2^*U_B^{(2)}}{\hat{s}_2 - \hat{s}_1}\right] \\
& - A_1\cos 2\theta_1\left[\frac{1}{2}V_{02YYY}^{(2)} + \mathrm{i}\hat{c}_1U_{02YY}^{(2)}\right] + A_1' - \mathrm{i}(Y + \gamma_1)\hat{c}_1A_1 - \frac{1}{2}\cos 2\theta_1A_1^*V_{20YYY}^{(3)} \\
& + \mathrm{i}\hat{c}_1A_1\big[U_{00YY}^{(2)} - 4\partial_{YYY}^3\big(U_1^{(1)}U_1^{(1)*}\big)\big] - \frac{2\mathrm{i}(\hat{c}_1^2\hat{s}_2^2 + \hat{s}_1^2\hat{c}_2^2)}{\hat{s}_2^2\hat{c}_1}A_1\partial_{YYY}^3\big(U_2^{(1)}U_2^{(1)*}\big),
\end{aligned}$$

while (24) has a solution

$$V_{1YY}^{(3a)} = -A_1 + 2\int_{-\infty}^{X} \big[A_1'(X_0) - \mathrm{i}\gamma_1 \hat{c}_1 A_1(X_0)\big] \mathrm{e}^{-\mathrm{i}\widetilde{Y}_1(X-X_0)}\,\mathrm{d}X_0$$

$$+ \int_{-\infty}^{X}\int_{-\infty}^{X_2}\int_{-\infty}^{X_1} \mathrm{d}X_0\,\mathrm{d}X_1\,\mathrm{d}X_2 \big[K_1^{(3a)} A_2(X_0) A_2^*(X_1) A_1(X_2)\,\mathrm{e}^{\mathrm{i}(\widetilde{Y}_2(X_0-X_1)+\widetilde{Y}_1(X_2-X))}$$

$$- K_1^{(3a)} A_2^*(X_0) A_2(X_1) A_1(X_2)\,\mathrm{e}^{\mathrm{i}(\widetilde{Y}_1(X_2-X)+\widetilde{Y}_2(X_1-X_0))}$$

$$+ K_1^{(3b)} A_1^*(X_0) A_1(X_1) A_1(X_2)\,\mathrm{e}^{\mathrm{i}\widetilde{Y}_1(X_1-X_0+X_2-X)}$$

$$+ K_1^{(3c)} A_1(X_0) A_1^*(X_1) A_1(X_2)\,\mathrm{e}^{\mathrm{i}\widetilde{Y}_1(X_0-X_1+X_2-X)}$$

$$+ K_1^{(3d)} A_1(X_0) A_1(X_1) A_1^*(X_2)\,\mathrm{e}^{\mathrm{i}\widetilde{Y}_1(X_0+X_1-X_2-X)}$$

$$+ K_1^{(3e)} A_2(X_0) A_1(X_1) A_2^*(X_2)\,\mathrm{e}^{\mathrm{i}(\widetilde{Y}_2(X_0-X_2)+\widetilde{Y}_1(X_1-X))}$$

$$+ K_1^{(3f)} A_1(X_0) A_2(X_1) A_2^*(X_2)\,\mathrm{e}^{\mathrm{i}(\widetilde{Y}_1(X_0-X)+\widetilde{Y}_2(X_1-X_2))}$$

$$+ K_1^{(3g)} A_2^*(X_0) A_1(X_1) A_2(X_2)\,\mathrm{e}^{\mathrm{i}(\widetilde{Y}_2(X_2-X_0)+\widetilde{Y}_1(X_1-X))}$$

$$+ K_1^{(3h)} A_1(X_0) A_2^*(X_1) A_2(X_2)\,\mathrm{e}^{\mathrm{i}(\widetilde{Y}_1(X_0-X)+\widetilde{Y}_2(X_2-X_1))}\big],$$

where the kernels are

$$K_1^{(3a)} = -2\hat{c}_2^5\hat{s}_2^2\big(2\hat{s}_1^2-1\big)\frac{(X_0-X_1)^3}{\hat{c}_1},$$

$$K_1^{(3b)} = 2\hat{c}_1^4\hat{s}_1^2\big(2\hat{s}_1^2-1\big)(X_0-X_1)\big[(2X_0-X_1-X_2)(X_0-X_1)+2\hat{s}_1^2(X_1-X_2)(X_0-X_2)\big],$$

$$K_1^{(3c)} = -2\hat{c}_1^4\hat{s}_1^2\big(2\hat{s}_1^2-1\big)(X_0-X_1)\big[2\hat{s}_1^2(X_1-X_2)(2X_1-X_0+X_2)+(X_0-X_2)(X_0-X_1)\big],$$

$$K_1^{(3d)} = 2\hat{c}_1^4\hat{s}_1^2\big(2\hat{s}_1^2-1\big)(2X_2-X_0-X_1)\big[4\hat{s}_1^2(X_1-X_2)(X_0-X_2)+(X_0-X_1)^2\big],$$

$$K_1^{(3e)} = \frac{2\hat{s}_2^2\hat{c}_2^2 L_1^{(3e)}}{\hat{c}_1},$$

$$\begin{aligned}
L_1^{(3e)} = {}& \hat{s}_1^2\big(\hat{c}_2^2+\hat{c}_1^2-4\hat{c}_1^2\hat{c}_2^2\big)(\hat{c}_1+\hat{c}_2)X_2^3+\hat{c}_2^3\big(2\hat{c}_1^2-1\big)X_0^3\\
&+X_2^2X_0\big[\hat{c}_1^3\hat{c}_2^2\big(3+4\hat{s}_1^2\big)-2\hat{c}_2^2\hat{c}_1-2\hat{c}_2\hat{c}_1^2\hat{s}_1^2+\hat{c}_2^3\big(4\hat{c}_1^2\big(1+2\hat{s}_1^2\big)-3\big)-\hat{c}_1^3\hat{s}_1^2\big]\\
&+\hat{c}_2X_0^2X_2\big[\hat{c}_2^2\big(3+4\hat{c}_1^4-9\hat{c}_1^2\big)+\hat{c}_1\hat{c}_2\big(1-2\hat{c}_1^2\big)+\hat{c}_1^2\hat{s}_1^2\big]+\hat{c}_1\hat{c}_2X_0^2X_1\big[\hat{c}_2^2\hat{c}_1\big(4\hat{s}_1^2-1\big)-\hat{c}_1\hat{s}_1^2+\hat{c}_2\big(2\hat{c}_1^2-1\big)\big]\\
&+\hat{c}_1^3X_1^2(X_0+X_2)\big(\hat{s}_1^2+4\hat{c}_1^2\hat{c}_2^2-3\hat{c}_2^2\big)+\hat{c}_1X_2^2X_1\big[\hat{c}_1\hat{s}_1^2\big(4\hat{c}_2^2-1\big)(2\hat{c}_1+\hat{c}_2)-\hat{c}_2^3\hat{c}_1-\hat{c}_2^2\big]\\
&+2\hat{c}_1X_0X_1X_2\big[\hat{s}_1^2(\hat{c}_1+\hat{c}_2)\big(\hat{c}_1+\hat{c}_2-4\hat{c}_2^2\hat{c}_1\big)+\hat{c}_1\hat{c}_2\big(\hat{c}_2^2-\hat{s}_1^2\big)\big],
\end{aligned}$$

$$K_1^{(3f)} = \frac{2\hat{s}_1^2\hat{c}_2^2 L_1^{(3f)}}{\hat{c}_2^2-\hat{c}_1^2},$$

$$\begin{aligned}
L_1^{(3f)} = {}& \hat{s}_2^4\hat{c}_1^3\big(3\hat{c}_2^2-1\big)(\hat{c}_2+\hat{c}_1)X_2^3+\hat{c}_1^2X_1^2X_2\big[\hat{c}_1^2\big(\hat{c}_2^3\big(7-3\hat{c}_2^2\big)(\hat{c}_1+\hat{c}_2)-\hat{c}_2^2-1\big)\\
&+\hat{c}_1\hat{c}_2\big(10\hat{c}_2^6-22\hat{c}_2^4+13\hat{c}_2^2-2\big)-4\hat{c}_2^4-2\hat{s}_2^2\hat{c}_2^6-3\hat{c}_1^3\hat{c}_2+2\hat{c}_2^2\big]\\
&+\hat{c}_2\hat{c}_1^2X_1^3\big[\big(3\hat{c}_1\hat{c}_2^5-7\hat{c}_1\hat{c}_2^3\big)(\hat{c}_1-\hat{c}_2)+3\hat{c}_1^2\hat{c}_2+\hat{c}_1+2\hat{c}_2^3-\hat{c}_2-5\hat{c}_1\hat{c}_2^2+\hat{c}_2^5\hat{s}_2^2\big]\\
&+X_2^2X_1\big[\hat{c}_2^6\big(8-3\hat{c}_1^4-19\hat{c}_1^2\big)+\hat{c}_1^5\big(3\hat{c}_2-7\hat{c}_2^3\big)+\hat{c}_1^3\big(2\hat{c}_2-8\hat{c}_2^3\big)\\
&+\hat{c}_1^4\big(2-7\hat{c}_2^2\hat{s}_2^4\big)+\hat{c}_2^5\hat{c}_1\big(8+3\hat{c}_2^2+16\hat{c}_1^2\big)-3\hat{c}_2^3\hat{c}_1-\hat{c}_1^2\hat{c}_2^2+\hat{c}_2^4\big(12\hat{c}_1^2-4\big)-8\hat{c}_1\hat{c}_2^7+9\hat{c}_2^8\hat{c}_1^2-4\hat{c}_2^8\big]\\
&+\hat{c}_1^2X_1^2X_0\big[\hat{c}_1\hat{c}_2^5\big(\hat{s}_2^2+3\hat{c}_1^2\big)+\big(2\hat{c}_2^3+\hat{c}_1\big)(\hat{c}_1-\hat{c}_2)\\
&+\hat{c}_1^3\hat{c}_2\big(3-7\hat{c}_2^2\big)+\hat{c}_2^2\big(2-9\hat{c}_1^2\big)+15\hat{c}_1^2\hat{c}_2^4-3\hat{c}_2^6\big(1+2\hat{c}_1^2\big)+2\hat{c}_2^8\big]\\
&+\hat{c}_1^3\hat{c}_2X_0^2X_1\big[\hat{c}_2^4\hat{s}_2^2-1+3\hat{c}_1^2\big(1+\hat{c}_2^4\big)+\hat{c}_2^2\big(2-7\hat{c}_1^2\big)\big]\\
&+\hat{c}_1X_2^2X_0\big[\hat{c}_1\hat{c}_2^2\big(5-8\hat{c}_2^2+6\hat{c}_2^4-2\hat{c}_2^6\big)+\hat{c}_2^3\hat{s}_2^4\big(4-9\hat{c}_1^2\big)+\hat{c}_1^4\hat{c}_2^3\big(3\hat{c}_2^2-4\big)\hat{c}_1^2\hat{c}_2\big(2-\hat{c}_2^4\big)-\hat{c}_1^3\big(1+\hat{c}_2^2\hat{s}_2^2\big)\big]\\
&+\hat{c}_2\hat{c}_1^2X_0^2X_2\big[\hat{c}_1\hat{c}_2^2\big(1+\hat{c}_2^2\hat{s}_2^2\big)+\hat{c}_2\hat{s}_2^4\big(1-2\hat{c}_2^2\big)+\hat{c}_1^3\big(2-6\hat{c}_2^2+3\hat{c}_2^4\big)\big]
\end{aligned}$$

$$+ 2\hat{c}_1 X_0 X_1 X_2 \big[\hat{c}_1 \hat{c}_2^2 \big(3\hat{c}_2^2 - 1 - \hat{c}_2^4\big) + \hat{c}_2^3 \hat{s}_2^4 \big(2\hat{c}_1^2 - 1\big)\big(-1 + 8\hat{c}_2^2 - 14\hat{c}_2^4 + 6\hat{c}_2^6\big)\big],$$

$$K_1^{(3g)} = \frac{2\hat{s}_2^2 \hat{c}_2^2 L_1^{(3g)}}{\hat{c}_1},$$

$$\begin{aligned}
L_1^{(3g)} = {} & 2\hat{c}_2^3\big(1 - 2\hat{c}_1^2\big)X_0^3 + \hat{s}_1^2(\hat{c}_2 - \hat{c}_1)\big(4\hat{c}_1^2\hat{c}_2^2 - \hat{c}_2^2 - \hat{c}_1^2\big)X_2^3 - \hat{c}_2 X_0^2 X_2\big[\hat{c}_2^2\big(3 - 9\hat{c}_1^2 + 4\hat{c}_1^4\big) - \hat{c}_1\hat{c}_2\hat{s}_1^2 + \hat{c}_1^2\hat{s}_1^2\big] \\
& + \hat{c}_1 X_2^2 X_1\big[2\hat{s}_1^2\hat{c}_1\big(\hat{c}_1\big(4\hat{c}_2^2 - 1\big) - 2\hat{c}_2^3\big) + \hat{c}_1\hat{c}_2 - \hat{c}_2^2\big] \\
& + X_2^2 X_0\big[\hat{c}_1^3\big(7\hat{c}_2^2 - \hat{s}_1^2\big) + \hat{c}_2^3\big(3 + 8\hat{c}_1^4 - 12\hat{c}_1^2\big) + 2\hat{c}_1^2\hat{s}_1^2\hat{c}_2 - 2\hat{c}_1\hat{c}_2^2\big(1 + 2\hat{c}_1^4\big)\big] \\
& - 2\hat{c}_1\hat{s}_1^2 X_0 X_1 X_2\big[\hat{c}_1\hat{c}_2\hat{s}_2^2 + \hat{c}_2^2\big(4\hat{c}_1^2 - 1\big) + 4\hat{c}_1\hat{c}_2^3 - \hat{c}_1^2\big] + \hat{c}_1\hat{c}_2 X_0^2 X_1\big[\big(4\hat{c}_1^3 - 3\hat{c}_1\big)\hat{c}_2^2 + \big(2\hat{c}_1^2 - 1\big)\hat{c}_2 + \hat{c}_1\hat{s}_1^2\big] \\
& + \hat{c}_1^3\big(\hat{s}_1^2 + 4\hat{c}_1^2\hat{c}_2^2 - 3\hat{c}_2^2\big)X_1^2(X_2 - X_0),
\end{aligned}$$

$$K_1^{(3h)} = \frac{2\hat{s}_1^2 \hat{c}_2^2 L_1^{(3h)}}{\hat{c}_1},$$

$$\begin{aligned}
L_1^{(3h)} = {} & \hat{c}_1^2 + \hat{c}_2\big(\hat{s}_2^2 - 3\hat{c}_1^2 + 4\hat{c}_1^2\hat{c}_2^2\big)X_1 X_0^2 + \hat{c}_1^2\hat{c}_2\big(3\hat{c}_1^2 - \hat{s}_2^2 - 4\hat{c}_1^2\hat{c}_2^2\big)X_2 X_0^2 \\
& + 2\hat{c}_1\big[\hat{c}_1^2 2\hat{c}_2^2\hat{s}_2^2\big(1 - 4\hat{c}_1^2\big)\big]X_1 X_2 X_0 + \hat{s}_2^2(\hat{c}_2 - \hat{c}_1)\big(4\hat{c}_1^2\hat{c}_2^2 - \hat{c}_2^2 - \hat{c}_1^2\big)X_2^3 \\
& + \hat{c}_2 X_1^3\big[\hat{c}_1^3\hat{c}_2 + \hat{s}_2^2\big(\hat{c}_1\hat{c}_2 + \hat{c}_1^2\hat{s}_2^2 + \hat{c}_2^2 - 4\hat{c}_1^3\hat{c}_2\big)\big] + \hat{c}_1 X_0 X_2^2\big[\hat{c}_1^3\big(4\hat{c}_2^3 - \hat{c}_2\big) - \hat{c}_1^2 + \hat{s}_2^2\big(2\hat{c}_2^2\big(4\hat{c}_1^2 - 1\big) + \hat{c}_2\hat{c}_1\big)\big] \\
& + \hat{c}_1 X_1^2 X_0\big[\hat{s}_2^2\hat{c}_2\big(4\hat{c}_1^2 - 1\big)(2\hat{c}_2 + \hat{c}_1) - \hat{c}_1^3\hat{c}_2 - \hat{c}_1^2\big] \\
& + X_1 X_2^2\big[\hat{c}_1^2\hat{s}_2^2\hat{c}_2\big(4\hat{c}_1^4 + 2\big) + \hat{c}_1^3\big(3\hat{c}_2^2 - 2\big) - \hat{c}_1^4\hat{c}_2 - \hat{s}_2^3\hat{c}_2(3\hat{c}_2 - \hat{c}_1)\big(4\hat{c}_1^2 - 1\big)\big] \\
& + X_2 X_1^2\big[\hat{s}_2^2\hat{c}_2\big(4\hat{c}_1^4 - 2\hat{c}_1^2 + 3\hat{c}_2^2\big(4\hat{c}_1^2 - 1\big)\big) - \hat{c}_1\hat{c}_2^2\hat{s}_2^2 + \hat{c}_1^3\big(\hat{c}_1\hat{c}_2 + 1 + \hat{c}_2^2 - 4\hat{c}_2^4\big)\big].
\end{aligned}$$

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